Newton's "derivation" of the inverse square law of gravity

From observations of the night sky, it was clear to Newton (and many before him) that there must be some form of attraction between the earth and the moon, and the sun and the planets that caused them to orbit around the Sun.

Yet, it was not at all clear that the same force of attraction could be responsible for the behavior of falling bodies near the surface of the earth.

Newton postulated that **one force called gravity** was responsible for both motions. His problem, then, was to attempt to determine the force law.

His method involved a comparison of the motions of the moon around the earth and an object (e.g., apple) falling towards the earth. Since neither the apple or the moon are moving in a straight line at constant speed, they must each be undergoing an acceleration.

Would it be possible to determine the nature of the force responsible for that acceleration? In order to do so, he must first determine the accelerations of the moon and the apple.

Know Facts about earth and moon

Fact -1: Measured. The acceleration of the apple is 9.8 m/s^2 or $a_A = 9.8$ m/s^2.

Fact-2: The radius of the earth is $R_A = 6.37 \times 10^6$ m.

Fact-3: Measured. The distance from the earth to the moon is $R_{moon} = 3.84 \times 10^8$ m.

Fact-4: Measured. The moon goes around the earth once in 27.3 days.

Calc-1: $C_{moon} = 2 \pi R_{moon} = 2 \times \pi \times (3.84 \times 10^8) = 2.413 \times 10^9$ m.

Calc-2: $v_{moon} = C_{moon} \div (27.3 \times 24 \times 60 \times 60) = 1019$ m/s.

Since the moon is in (nearly) circular orbit around the earth, its motion can be approximated as circular. Therefore, the acceleration of the moon is a centripetal acceleration whose formula was discovered by Christian Huygen's in 1657.

$$a_{moon} = v^2_{moon} / R_{moon}$$

And that value is

Calc-3: $a_{moon} = v^2_{moon} / R_{moon} = 1019^2 / 3.84 \times 10^8 = 2.704 \times 10^{-3}$ m/s^2

Forming the acceleration and radius ratios

We now have measurements for the accelerations of the moon and the apple.

Newton postulated that the attractive force between the earth and the apple or between the earth and the moon should depend upon the distance measured from the center of earth to the center of the apple, or the center of the earth to the center of the moon.

We already know the distance from the center of earth to the center of the moon – it is given above by R_{moon}. The distance from the center of the earth to the center of the apple is just the radius of the earth $\mathbf{R_A}$ (ignoring the radius of the apple).

Apple	$a_A = 9.8\ m/s^2$	$R_A = 6.37 \times 10^6\ m$
Moon	$a_{moon} = 2.704 \times 10^{-3}\ m/s^2$	$R_{moon} = 3.84 \times 10^8\ m$

Notice that as R, the distance between the objects, increases, that the acceleration "a" caused by the earth's gravity decreases. This implies an inverse relationship of some kind.

$$\frac{a_{moon}}{a_{apple}} \propto \frac{R_{apple}}{R_{moon}}$$

note that apple/moon are reciprocal on either side of the proportionality.

Calculating the earth/apple and earth/moon ratios

But, we and Newton find that the two ratios are NOT equal

$$\frac{a_{moon}}{a_{apple}} = 2.76e^{-4} \quad \propto \quad \frac{R_{apple}}{R_{moon}} = 1.66e^{-2} \quad ?\ NO.$$

But, if we square the distance term we get what Newton called 'pretty nearly' the same value as the acceleration ratio

$$\frac{a_{moon}}{a_{apple}} = 2.76e^{-4} \quad \propto \left(\frac{R_{apple}}{R_{moon}}\right)^2 = 2.75e^{-4} \quad ?YES.$$

Thus, in 1678 Newton concluded that gravity varied between earth and the apple and the earth and the moon as an inverse square law!

$$a_g = \frac{k}{r^2} \quad \text{where k = G * m.}$$

The gravitational constant was not measured until1789 with a very difficult experiment.

Note this calculation is approximate because the moon does not orbit around the center of the earth, but the earth and moon rotate around the earth/moon center of mass which is 1000 km below the earth's surface.

Other supporting evidence for Newton's gravity law

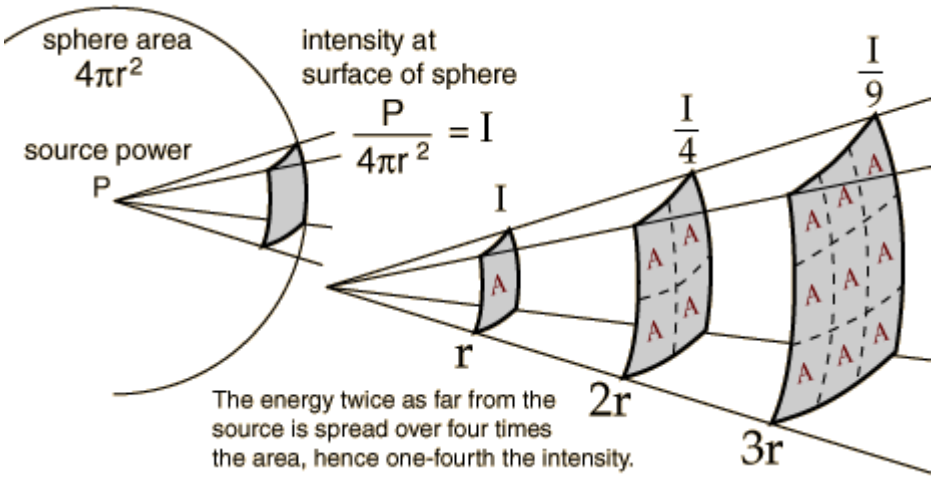
This inverse square law result is consistent with Kepler's third law found in 1610: the square of the orbital period (T) equals the cube of the orbital radius (R).

Newton explained the tides as a result of the earth and sun gravitational pull.

Newton showed that with an inverse square law for gravity, the stable orbits around a mass (e.g., Sun) would be conic section (ellipse, parabola, hyperbola) trajectories. This explained comet trajectories.

An inverse square law is required by any field that spreads uniformly in space from a point if the field strength is finite and extends to infinity. The electric field (Coulomb's law), gravity field, E-M radiation, sound and light all obey inverse square laws.

How to further test the inverse square law given there are planets orbiting around the sun?



Historical context

250 BC. Aristotle says the earth is the center of universe and that the sun, moon, planets go around stationary non-rotating earth in the rotating 'crystal spheres' via *a primum mobile*. Church incorporates this pagan physics into their dogma. Aristotle does not understand inertia and wrote incorrectly that bigger masses fall faster and constant velocity motion requires constant push (force).

1554. Copernicus publishes book saying the sun is the center of the solar system and the earth spins once a day and the moon orbits the earth.

1600. Giordano Bruno burned at stake for saying the stars are suns and earth rotates around sun: heliocentrism.

1610. Kepler calculates for eight years (using new logarithm tables) and defines three laws from new accurate data from Tyco Brahe and shows Copernicus is correct.

1611. Galileo makes a telescope and views the phases of Venus and the moon's of Jupiter. Also, first to elucidate the inertia of mass with inclined plane and ball-dropping. He is jailed for violating church dogma by saying that the sun is center of solar system. He does not understand gravity though.

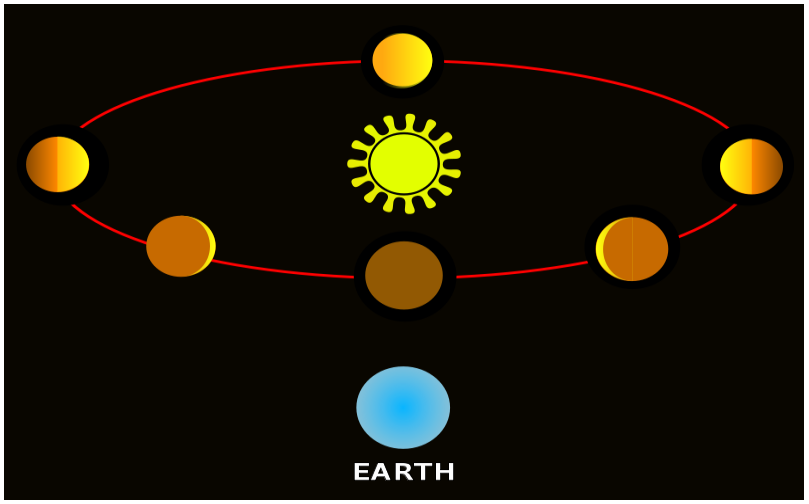
1630. Descartes sees what happened to Galileo and retracts his book to avoid conflict with the church. Develops Cartesian coordinate system and marries algebra and geometry.

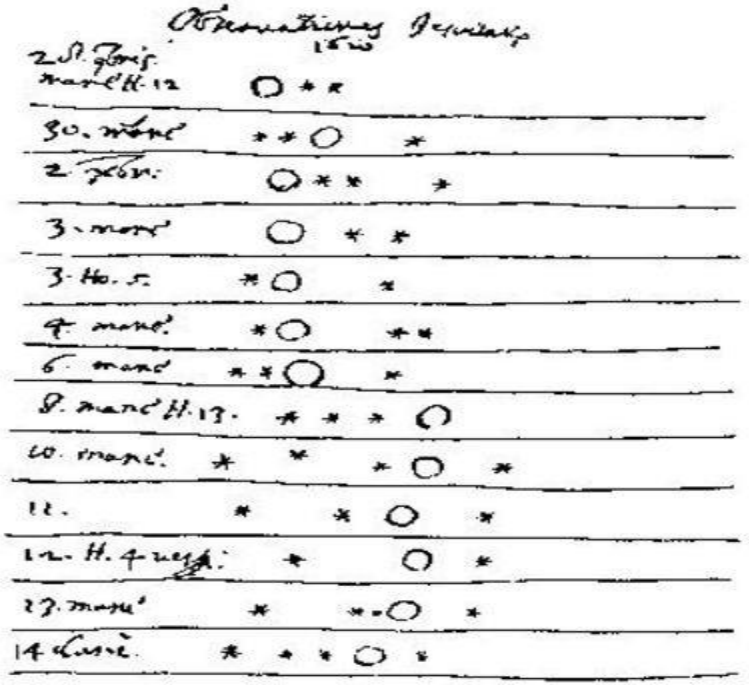
1678. Newton provides derivation of gravitational force and how the planets move! And starts modern science with calculus, mechanics, and light research. Doesn't know what gravity is though.

1846. Neptune planet discovered by anomalies of Uranus orbit using Newton's gravity law.

1851. Foucault pendulum experiment proves that earth is rotating. Church finally relents.

Aristotle and Church versus Galileo





Galileo at end of Inquisition in 1633 after being forced under pain of death to recant his theory that the earth moves around the sun:

" and yet it moves".